Problem of the Week 1

2 9 □ □

A four digit number has been written on a piece of paper. The last two digits were then blotted out as shown.

The four digit number is divisible by five and eleven, but not by two.

What four digit number is it?

2915

As the number is divisible by 5, its units digit digit must be 0 or 5.

As the number not divisible by 2, it is not even. Hence its last digit is 5. So the number is $29x5$.

The test for a four-digit number to be divisible by 11 is that the sum of its hundreds and units digits equals the sum of its tens and thousands digits or differs from this sum by a multiple of 11.

Since $9 + 5 = 14$, the only possibility is that $(9 + 5) - (2 + x) = 11$. Therefore $12 - x = 11$. Hence $x = 1$. So the number is 2915.

Problem of the Week 2

$$2, 3, 5, 10, .., .., .., .., .., x$$

In this sequence each number after the second is the sum of all the previous numbers in the sequence.

What is the 10th number in the sequence?

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640

Each number from the 3rd number onwards is double the previous number.

We prove this as follows. Suppose $k \geq 2$ and and the sum of the first k numbers is n. Then the $(k+1)$th number in the sequence is n.

Therefore the sum of the first $k+1$ numbers in the sequence is $n + n = 2n$. Hence this is the $(k+2)$th number in the sequence.

It follows that the first ten numbers in the sequence are $2, 3, 5, 10, 20, 40, 80, 160, 320, 640$.

Problem of the Week 3

	*			A
		B		
D		C		
			E	

I want to fill in this grid so that each of the letters A, B, C, D, and E occurs once in each row, once in each column and once in each of the two main diagonals.

Which letter should go in the square marked with the asterisk?

D

The letter in square 1 is on the same diagonal as B, in the same row as C and D, and in the same column as E. Hence the letter in this square is A.

	D			A
		B		
D	[2]E	C	[1]A	
[3]A		[4]D	E	[5]C

The letter in square 2 is in the in the same row as A, C and D, and in the same diagonal as B, and hence is E.

There is an A in the top-right to bottom-left diagonal. Because there is an A in row 2 and column 5, this A is not in the top two squares on this diagonal. Hence it is in square 3.

It may now be seen that squares 4 and 5 contain D and C, respectively. It then follows that the asterisk needs to be replaced by D. (It is left to the reader to fill in the remaining squares.)

Problem of the Week 4

Dilly is 7 years younger than Dally. In 4 years time she will be half Dally's age.

What is the sum of their ages now?

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Dilly is 7 years younger than Dally. Therefore when Dilly is half Dally's age, Dilly will be 7 and Dally will be 14.

This will be in 4 years time. So now Dilly is $7 - 4 = 3$ and Dally is $14 - 4 = 10$.

$3 + 10 = 13$.

Problem of the Week 5

$$
\begin{array}{r}
fly \\
+\ fly \\
+\ fly \\
\hline
away \\
\hline
\end{array}
$$

In this addition sum each different letter represents a different non-zero digit.

What is the value of $a + w + a + y$?

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From the units column, we see that $3y = y +$ possibly a non-zero multiple of 10. Since $y \leq 9,\ 3y = y$ or $3y = y + 10$. Since $y \neq 0,\ 3y \neq y$. Therefore $3y = y + 10$. Hence $y = 5$ and there is a carry of 1 to the tens column. Since $fly < 1000, away < 3000$. So a is 1 or 2.

Including the carry of 1 from the units column, the total of the tens column is $3l + 1$ The only single-digit value of l for which a is 1 or 2 is $l = 7,$ giving $3l + 1 = 22$. Therefore $l = 7, a = 2$ and there is carry of 2 to the hundreds column.

Since $a = 2, away \geq 2000$ and hence $fly \geq 600$. It follows that $f \geq 6$. If $f = 6,$ then $w = 0$. Since $l = 7,\ f \neq 7$. If $f = 9,$ then $w = 9 = f$. Hence we conclude that $f = 8$. Therefore $w = 6$. Hence the sum is as shown on the right. So $a + w + a + y = 2 + 6 + 2 + 5 = 15$.

$$\begin{array}{r} 8\,7\,5 \\ +8\,7\,5 \\ +8\,7\,5 \\ \hline 2\,6\,2\,5 \\ \hline \end{array}$$

Problem of the Week 6

In California, a bottle of orange juice costs $3, but when you return the bottle, you get $2 back.

What is the largest number of bottles of juice you can buy if you start with $10?

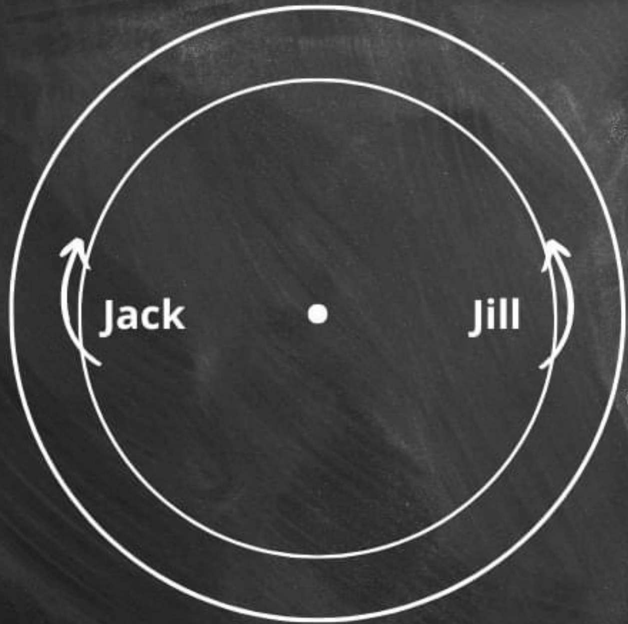
8

When you buy a bottle of orange juice for \$3, and receive \$2 back when you return the empty bottle you end up with \$1 less than you began with.
You can go on buying bottles until you have less than \$3 left. So you can buy 8 bottles before you no longer have enough money to buy another bottle.

Problem of the Week 7

Jack dances clockwise round the Maypole, making one revolution every 5 seconds.

Starting from a point diametrically opposite Jack's starting point, Jill dances anticlockwise, making one revolution every 6 seconds.



How many times do they pass each other in the first minute?

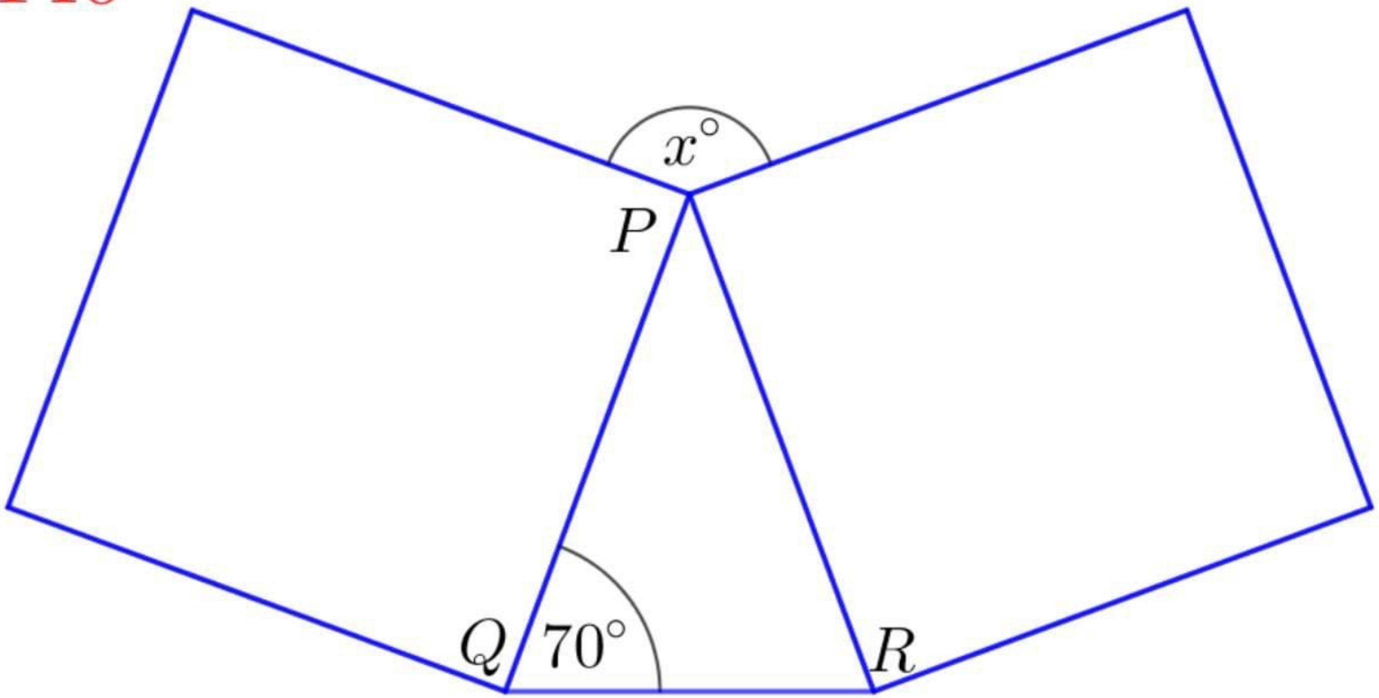
22

The answer depends on the *relative* positions of Jack and Jill.
Jack makes one revolution every 5 seconds. So he goes completely round the Maypole clockwise $\frac{60}{5} = 12$ times each minute.
Jill makes one revolution every 6 seconds. So she goes completely round the Maypole anticlockwise $\frac{60}{6} = 10$ times each minute.
Therefore, relative to Jill, Jack does $12 + 10 = 22$ complete revolutions each minute.
It follows that they pass each other 22 times in the first minute.

Problem of the Week 8

The diagram shows two congruent squares.
What is the value of x ?

140



Because the squares are congruent, $PQ = PR$.
Therefore the triangle PQR is isosceles.
Therefore $\angle PRQ = \angle PQR = 70^\circ$.
Since the sum of the angles in a triangle is 180°,
$\angle RPQ = 180^\circ - 70^\circ - 70^\circ = 40^\circ$.
The sum of the angles at P is 360°, and the angles in the squares are each 90°. Therefore
$x^\circ = 360^\circ - (90^\circ + 40^\circ + 90^\circ) = 140^\circ$. Hence $x = 140$.

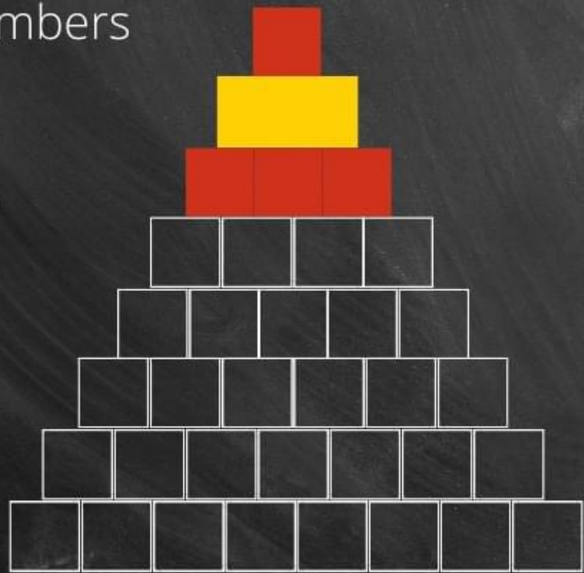
Problem of the Week 9

Donna is building a tower using coloured bricks.

She has 36 bricks, with equal numbers of red, yellow, and blue bricks.

In each row all the bricks are the same colour.

The top three rows are as shown. What are the colours of the bricks in the bottom 5 rows?



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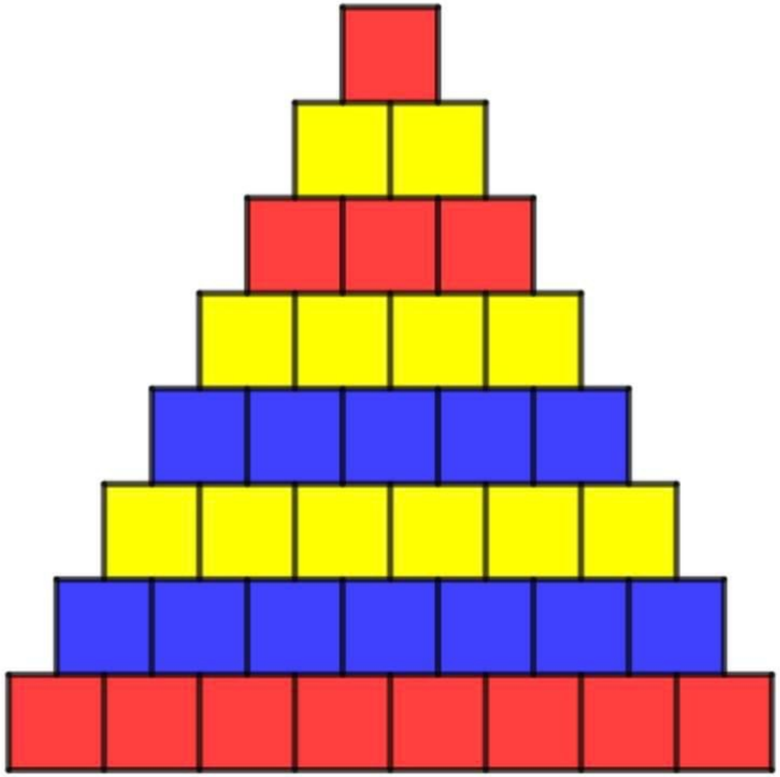
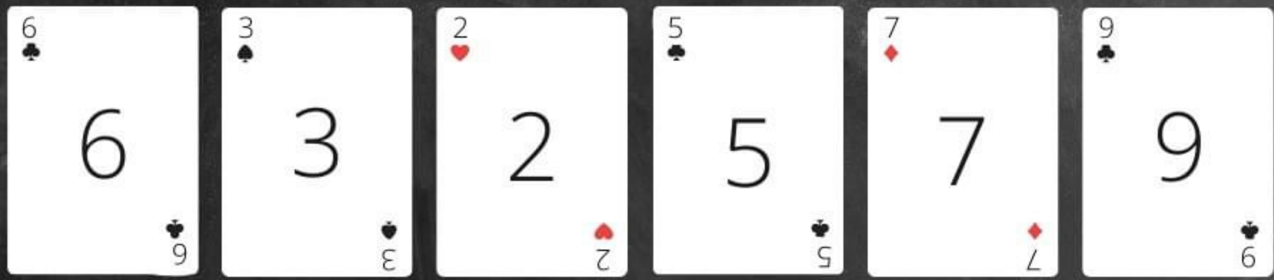


Donna has 12 red, 12 yellow and 12 blue bricks. After using 4 red bricks in the top three rows, 8 red bricks remain. The only way to place these so that all the bricks in the same row are the same colour is to put them all in the bottom row. This leaves rows of 4, 5, 6 and 7 for the remaining 10 yellow bricks and the 12 blue bricks. The 10 yellow bricks have to go in the rows of 4 and 6 bricks leaving the rows of 5 and 7 bricks for the 12 blue bricks, as shown above.

Problem of the Week 10



The six cards shown display the number 632579.

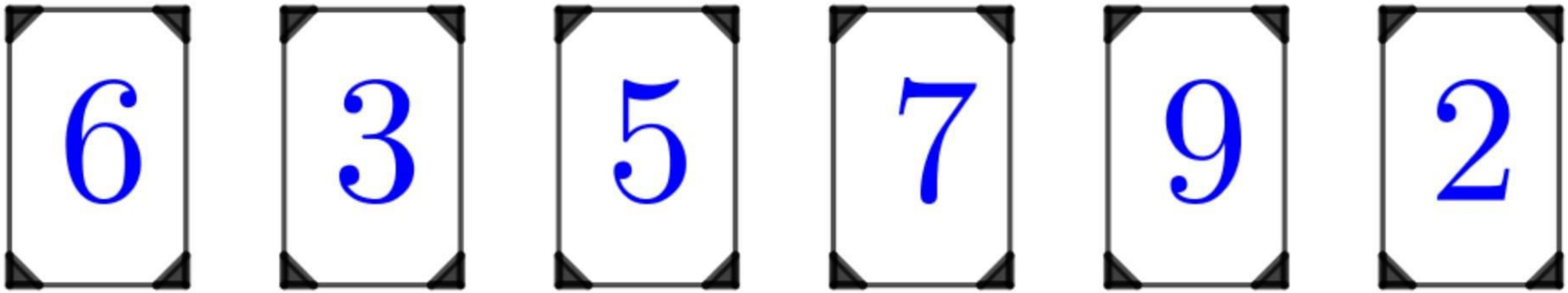
A *move* consists of swapping two adjacent cards. For example, after one move the number displayed could be 623579.

What is the least number of moves needed so that the number displayed changes from 632579 to a multiple of 4?

3

A number is a multiple of 4 provided that its last two digits (the units and tens digits) make up a number that is a multiple of 4.

By making the three swaps $2 \leftrightarrow 5$, $2 \leftrightarrow 7$ and $2 \leftrightarrow 9$, in this order, the display becomes



and 635792 is a multiple of 4.

Problem of the Week 11

5p, 2p, and 1p coins (or a mixture of any or all of these) are used to make a total of 11p.

In how many different ways can this be done?

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We list, systematically, all the ways to make a total of 11p using 5p, 2p and 1p coins.

(1) $2 \times 5\text{p} + 1 \times 1\text{p}$

(2) $1 \times 5\text{p} + 3 \times 2\text{p}$

(3) $1 \times 5\text{p} + 2 \times 2\text{p} + 2 \times 1\text{p}$

(4) $1 \times 5\text{p} + 1 \times 2\text{p} + 4 \times 1\text{p}$

(5) $1 \times 5\text{p} + 6 \times 1\text{p}$

(6) $5 \times 2\text{p} + 1 \times 1\text{p}$

(7) $4 \times 2\text{p} + 3 \times 1\text{p}$

(8) $3 \times 2\text{p} + 5 \times 1\text{p}$

(9) $2 \times 2\text{p} + 7 \times 1\text{p}$

(10) $1 \times 2\text{p} + 9 \times 1\text{p}$

(11) $11 \times 1\text{p}$

Problem of the Week 12

The diagram shows a square whose sides have length 1, a circle which goes through the vertices of the square, and four semicircles, each having one side of the square as its diameter.

The region coloured red is made up of all the points inside the semicircles but outside the circle.

What is the area of this region?

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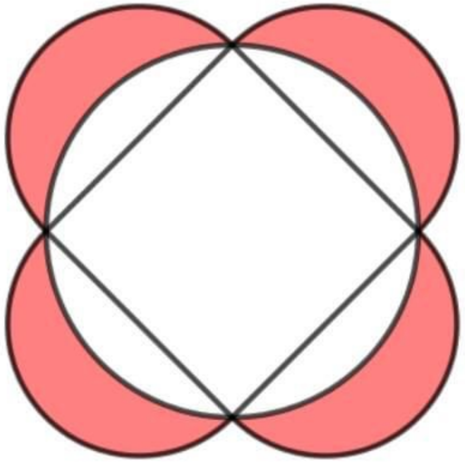
1



The area of the region coloured red is the area inside the the square and the four semicircles, but outside the circle.

The square has sides of length 1 and hence area $1^2 = 1$.

Each semicircle has radius $\frac{1}{2}$. Hence the total area of the four semicircircles is $4 \times (\pi(\frac{1}{2})^2) = \pi$.

By Pythagoras' Theorem, the length of the diagonals of the square is $\sqrt{2}$. Hence the area of the circle is $\pi(\frac{1}{2}\sqrt{2})^2 = \pi$.

Therefore the area of the coloured region is $1 + \pi - \pi = 1$.

Problem of the Week 13

A circle with circumference 1 rolls without slipping on the outside of an equilateral triangle with sides of length 1, as shown.

The circle rolls once around the triangle.

How many complete turns about it's centre does the circle make?

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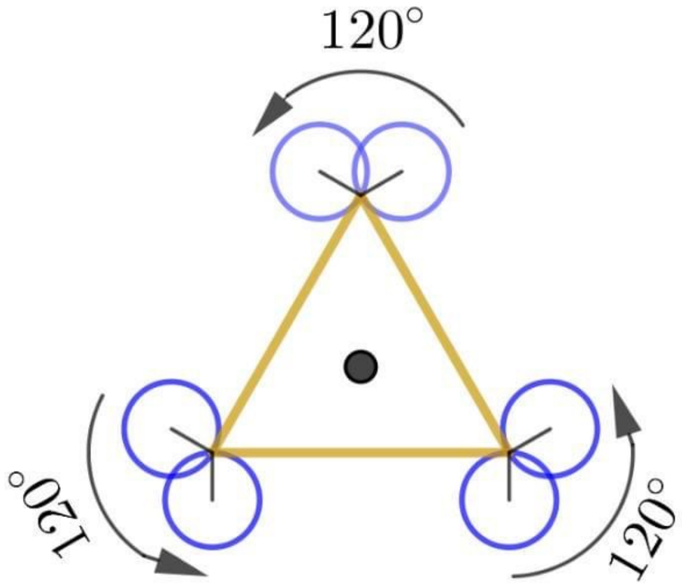
4



The circumference of the circle is equal to the length of each side of the triangle. Therefore the circle makes one complete turn as it rolls along each edge.
This makes a total of 3 complete turns.

In addition the circle turns through 120° at each vertex. These turns add up to 360°, another complete turn.

Therefore, in total, the circle makes 4 complete turns.

Problem of the Week 14

I bought a camera for £60 in the sale (20% off everything).

What was the full cost of the camera before the sale?

£75

A 20% reduction means $\dfrac{1}{5}$th off. So £60 was $\dfrac{4}{5}$ths of the full price of the camera.

Hence the full price was $\dfrac{5}{4} \times £60 = £75$.

Problem of the Week 15

$$\begin{array}{rcc} & p & q \\ \times & & r \\ \hline & s & t \\ \hline \end{array}$$

In the multiplication sum above, the letters *p*, *q*, *r*, *s*, and *t* represent the digits 1, 2, 3, 4, and 5 in some order.

Which number is the answer to this sum?

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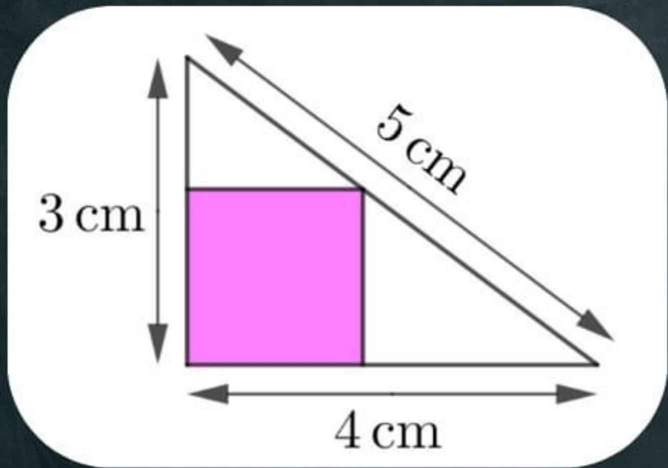
The product $q \times r$ is a number with units digit t. The only possibilities are $3 \times 4 = 12$ or $4 \times 3 = 12$. In either case p and s are 1 and 5, in some order. If $p = 5$ the answer will have more than two digits. So $s = 5$. Therefore the sum is

$$\begin{array}{r} 1\ 3 \\ \times \quad 4 \\ \hline 5\ 2 \\ \hline \end{array}$$

with the answer 52.

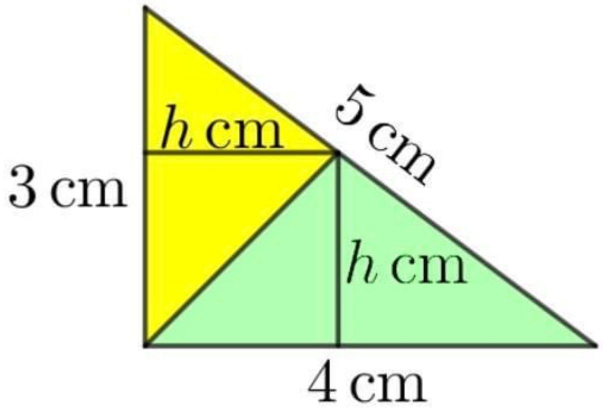
Problem of the Week 16



The diagram shows a square inside a right-angled triangle.

What fraction of the area of the triangle is taken up by the square?

$\frac{24}{49}$



We suppose that the sides of square have length h cm. We use the formula $\frac{1}{2}(\text{base} \times \text{height})$ for the area of a triangle. We deduce that the triangle has area $\frac{1}{2}(4 \times 3)\,\text{cm}^2 = 6\,\text{cm}^2$.

We can split the triangle into two smaller triangles, shown in yellow and green. Their areas are $\frac{1}{2}(3h)\,\text{cm}^2$ and $\frac{1}{2}(4h)\,\text{cm}^2$. It follows that $\frac{1}{2}(3h) + \frac{1}{2}(4h) = 6$. Therefore $h = \frac{12}{7}$. Hence the area of the square is $\left(\frac{12}{7}\right)^2\text{cm}^2$. As a fraction of the area of the triangle, this is $\dfrac{\left(\frac{12}{7}\right)^2}{6} = \dfrac{24}{49}$.

Problem of the Week 17

The Cecily Cardew School

THE IMPORTANCE OF BEING EARNEST

by Oscar Wilde

Friday, December 12th and
Saturday, December 13th 1997
at 7:30pm

Tickets: Adults £3, Children £1

Tickets for the school play cost £3 for adults, and £1 for children.
There were 600 tickets on sale for each performance.
On the Friday, although not all the tickets were sold, sales totalled £1320.

What is the smallest number of adult tickets that could have been sold?

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361

Suppose that a adults' tickets and c children's tickets were sold on the Friday. Then, because the total sales came to £1320, it follows that $3a + c = 1320$.

Since not all the tickets were sold, $a + c < 600$.

Since $3a + c = 1320$, we deduce that $2a = 1320 - (a + c)$.

Now, as $a + c < 600$, we have $2a > 1320 - 600 = 720$.

Hence, because $2a$ is even, the smallest possible value of $2a$ is 722.

Since $2a \geq 722$ implies $a \geq 361$, the smallest number of adult tickets that could have been sold is 361.

Problem of the Week 18

How many times between midday and midnight is the hour hand of a clock at right angles to the minute hand?

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The hour hand is at right angles to the minute hand twice each hour, with two exceptions. Since the hands are at right angles at 3pm, they are at right angles only three times between 2pm and 4pm, at just after 2:27pm, at 3pm and at just after 3:32pm. Similarly, because they are at right angles at 9pm, the hands are at right angles only three times between 8pm and 10pm.

Therefore the total number of times that the hands are at right angles between midday and midnight is $2 \times 12 - 2 = 22$.

Problem of the Week 19

A transport company's vans can each carry a maximum load of 12 tonnes.
A firm needs to deliver 24 crates each weighing 5 tonnes.

How many van loads will be needed to do this?

12

A van load cannot weigh more than 12 tonnes, and so cannot consist of more than 2 crates weighing 5 tonnes.

Therefore to carry 24 crates, $24 \div 2 = 12$ van loads will be needed.

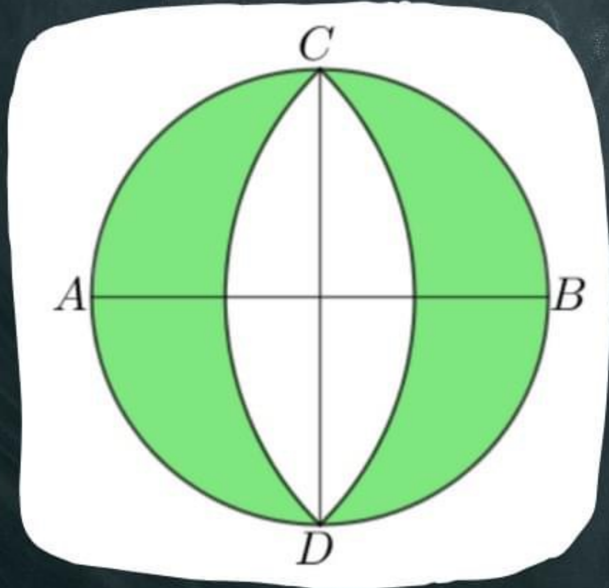
Problem of the Week 20

A B is a diameter of the circle which has radius 1cm.

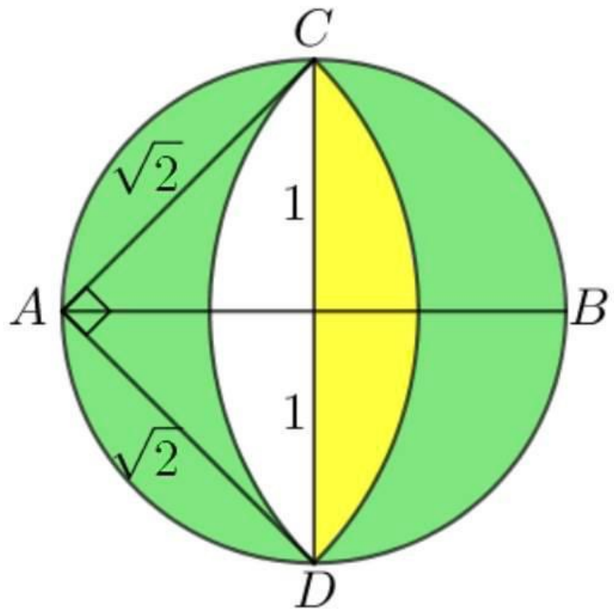
Two circular arcs of equal radius are drawn with centres *A* and *B*.

These arcs meet at the points *C* and *D* on the circle, as shown. *C D* is also a diameter of the circle.

What is the area of the region shaded green?



$2\ \text{cm}^2$



Because CD is a diameter of the circle, in the triangle CAD there is a right angle at A. Also, $AC = AD$.

Therefore by Pythagoras' Theorem, $AC^2 + AC^2 = CD^2 = 4\ \text{cm}^2$.

It follows that $AC^2 = 2\ \text{cm}^2$ and so $AC = \sqrt{2}\ \text{cm}$.

The area of the region shaded in yellow is the area of the quarter circle with radius AC less the area of the triangle CAD.

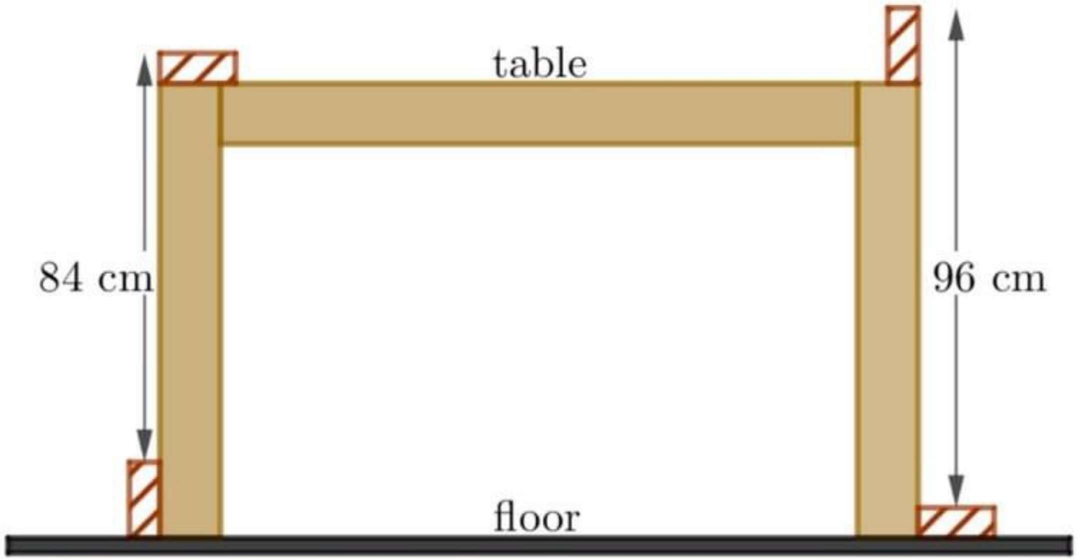
Hence its area is $\left(\frac{1}{4}\pi\sqrt{2}^2 - \frac{1}{2}(\sqrt{2}\times\sqrt{2})\right)\text{cm}^2 = (\frac{1}{2}\pi - 1)\ \text{cm}^2$.

The area of the region shaded green is the area of the circle, less twice the area of the region shaded yellow. Hence this area is $\left(\pi 1^2 - 2(\frac{1}{2}\pi - 1)\right)\ \text{cm}^2 = \left(\pi - (\pi - 2)\right)\ \text{cm}^2 = 2\ \text{cm}^2$.

Problem of the Week 21

Four identical blocks of wood are placed touching a table in the positions shown in the diagram.

How high is the table?



90 cm

Let the height of the table be h cm. Let the bricks have height a cm and width b cm.

From the information given in the diagram we have

$$h - a + b = 84$$

and

$$h + a - b = 96.$$

By adding these two equations, we obtain

$$2h = 180.$$

Therefore

$$h = 90.$$

Problem of the Week 22

The Queen of Hearts has lost her tarts!
She is sure that knaves who have not eaten the tarts will tell the truth and that any guilty knaves will tell lies.

When questioned, the knaves answer as follows:

Knave 1 - "One of us ate them."
Knave 2 - "Two of us ate them."
Knave 3 - "Three of us ate them."
Knave 4 - "Four of us ate them."
Knave 5 - "All five of us ate them."

Which knaves ate the tarts?

Knaves 1, 2, 3 and 5

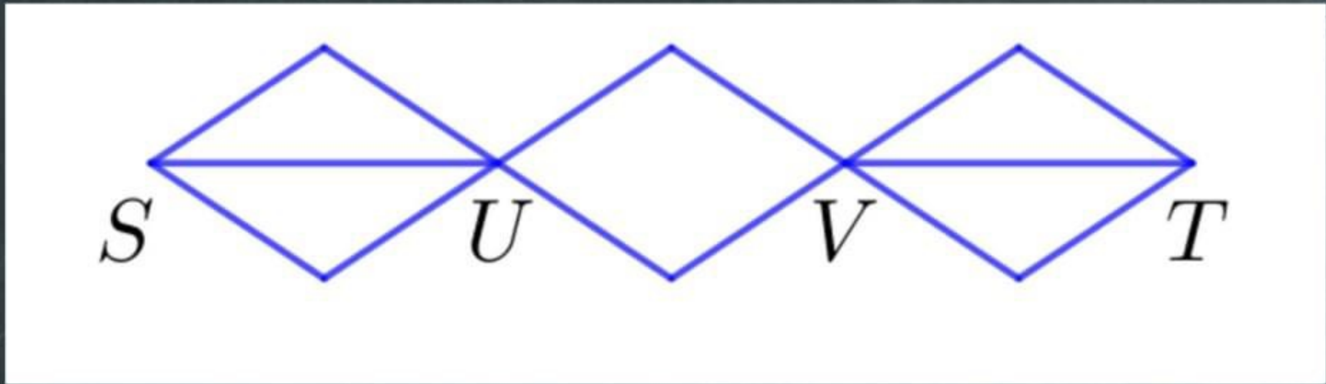
The Knaves contradict each other, so at at most one of them is telling the truth.

If none of them is telling the truth, they all ate the tarts. So Knave 5 is telling the truth. This contradiction shows they are not all lying.

Hence exactly one knave is telling the truth and four of them are lying. So four knaves ate the tarts. Hence Knaves 1, 2, 3 and 5 are lying. So these are the knaves who ate the tarts.

Problem of the Week 23

How many different routes are there from *S* to *T* that don't go through either of the points *U* and *V* more than once?



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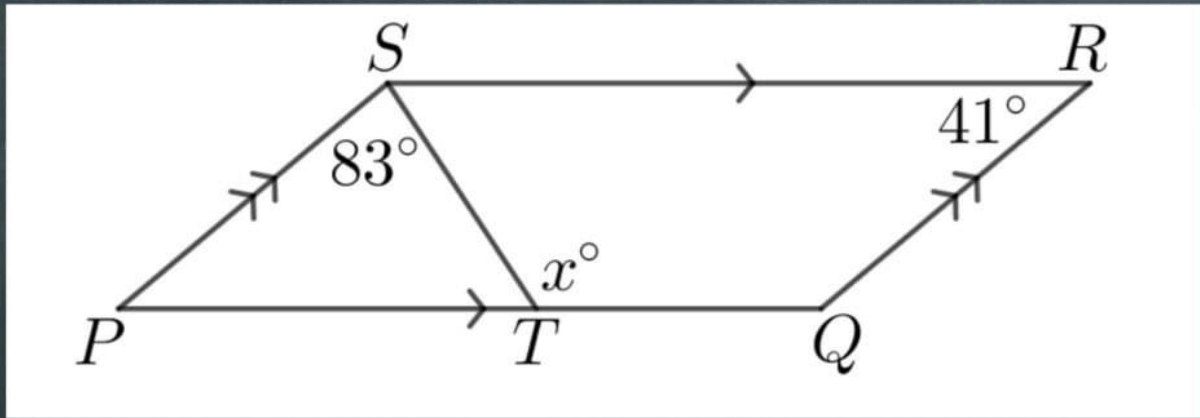
There are 3 routes from S to U, 2 routes from U to V, and 3 routes from V to T.

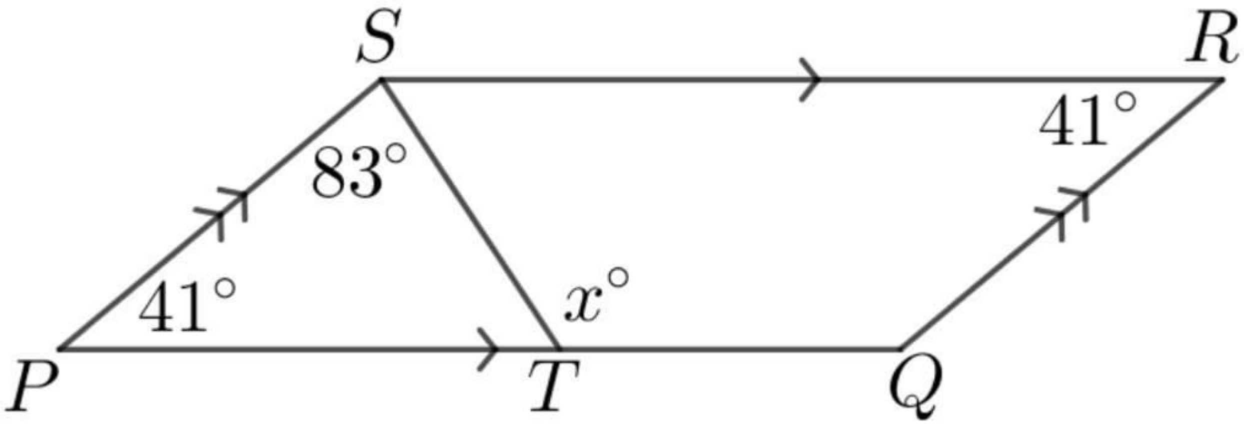
Therefore, as these routes may be chosen independently of each other, the total number of different routes is $3 \times 2 \times 3 = 18$.

Problem of the Week 24



PQRS **is a parallelogram.**
What is the angle $\angle STQ$?

124°



The opposite angles of a parallelogram are equal. Therefore $\angle TPS = \angle SRQ = 41^\circ$. Hence, by the External Angle Theorem, $\angle STQ = \angle TPS + \angle PST = 41^\circ + 83^\circ = 124^\circ$.

$Note:$ The External Angle Theorem is the theorem that says that each external angle of a triangle is the sum of the two opposite internal angles.

Problem of the Week 25

In the Soft Boulder Café each table has three legs, each chair has four legs and all the customers and the three members of staff have two legs each.

There are four chairs at each table.

At a certain time, three-quarters of the chairs are occupied by customers and there are 206 legs altogether in the café.

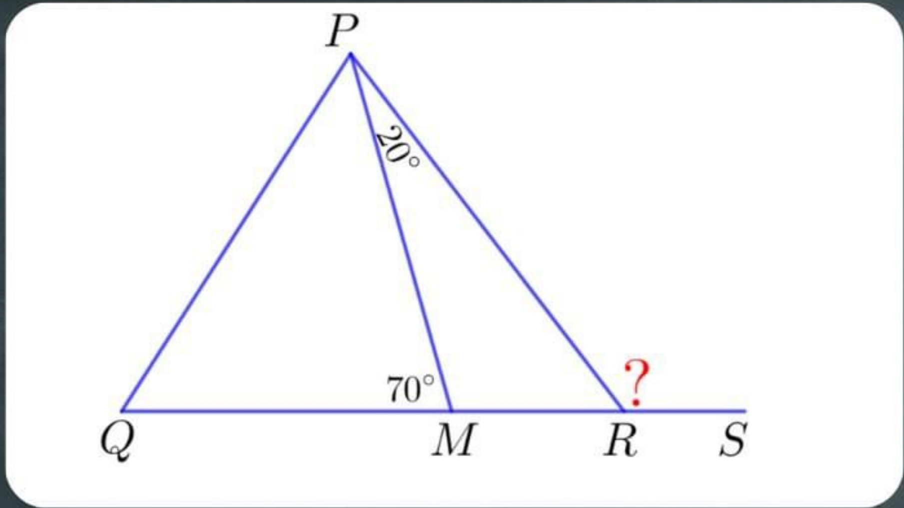
How many *chairs* does the café have?

32

Of the 206 legs, 6 belong to the members of staff, leaving 200 other legs. When three-quarters of the chairs are occupied, at an average table there are three customers with 6 legs between them. There are also the 3 legs of the table and the 16 legs of the 4 chairs. This makes an average of $6 + 3 + 16 = 25$ legs per table. Hence there are $200 \div 25 = 8$ tables

Therefore there are $8 \times 4 = 32$ chairs.

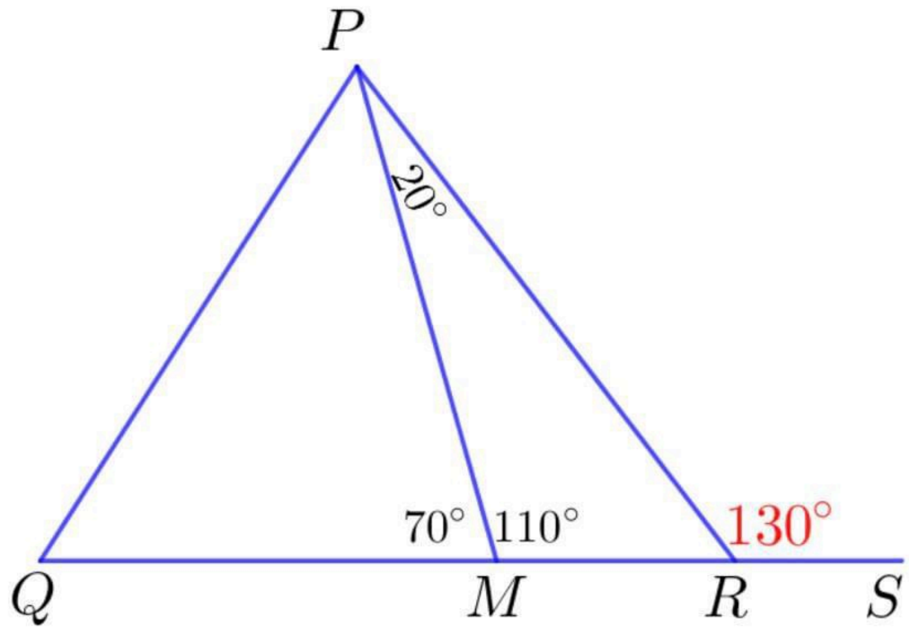
Problem of the Week 26



In the diagram $\angle RPM = 20°$ and $\angle QMP = 70°$.

What is $\angle PRS$?

130°



Because angles on a straight line add up to 180°, $\angle RMP + \angle QMP = 180°$. Therefore, $\angle RMP = 180° - \angle QMP = 180° - 70° = 110°$.

Therefore, by the External Angle Theorem, $\angle PRS = \angle RPM + \angle RMP = 20° + 110° = 130°$.

[Alternatively, because the angles in a triangle add up to 180°, $\angle PRM = 180° - (20° + 110°) = 50°$. Then, as angles on a line add up to 180°, $\angle PRS = 180° - 50° = 130°$.]